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Sensitivity Analysis for Fuzzy Linear Programming with its Applications in the Transportation Problem

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ABSTRACT

Fuzzy Linear programming provides the flexibility that decision maker needs in solving real life optimization problems. However even after formulating the problem, we cannot be sure that all values entered will remain the same for a long time. The role of sensitivity analysis is to study how the changes in the model affect the optimal solution. In this paper we study some cases of these changes and then apply our study on the famous transportation problem which we find a suitable engineering application. A numerical example is solved.

Key words: Fuzzy Sets; Fuzzy Numbers; Fuzzy Linear Programming; Transportation Problem.

Introduction

Optimization models assume that the data are precisely known and that the constraints represent a crisp set of feasible solutions. However in real life problems such assumptions are not always true. Bellman and Zadeh (1970) pointed out that in a fuzzy environment goals and constraints have the same nature and can be represented by a fuzzy set on a set of alternatives that contains the solution of a given multicriteria optimization problem. Linear programming is considered as a special kind of decision making problems. There exists some uncertainty in the values of parameters required in modeling a linear programming problem. Fuzzy linear programming (FLP) provides the flexibility in these uncertain values. Zimmermann (1976) fuzzified a linear programming problem as the fuzzy numbers became the source of flexibility. He also presented a fuzzy approach to multi-objective linear programming problem. But even after formulating the problem, one cannot be sure that all the values will remain the same for a long time. Also it is possible that wrong values may be entered. As time passes factors like cost, time or availability of products etc. change widely. Then we need to apply sensitivity analysis to the FLP problem.

Sensitivity analysis is an important tool in studying perturbations in any optimization problem. Tanaka and Asai (1984) proposed a method of allocating the given investigation cost to each fuzzy coefficients by using sensitivity analysis. Tanaka et al. (1986) formulated a FLP problem with fuzzy coefficient and the value of information was discussed via sensitivity analysis. Sakawa and Yano (1988) presented a fuzzy approach for solving multi-objective linear fractional programming problem using sensitivity analysis. Fuller (1989) proposed that the solution of FLP problems with symmetrical triangular fuzzy numbers is stable with respect to small changes in centers of fuzzy numbers. Verdegay and Aguado (1993) stated that whether or not a fuzzy optimal solution has been obtained using linear membership functions modeling the constraints, possible changes in these membership functions do not affect the former optimal solution. Gupta and Bhatia (2001) studied the measurement of sensitivity for changes of violations in the aspiration level for the fuzzy multi-objective linear fractional programming problem. Kheirfam and Hassani (2010) proposed a method for the sensitivity analysis for FLP problems with fuzzy variables. Kumar and Bhatia (2011) studied some important cases that are not considered by Kheirfam and Hassani (2010). Kumar and Bhatia (2012) treated fuzzy variable linear programming using ranking functions.

Preliminaries:

Some basic definitions and arithmetic operations on fuzzy numbers are reviewed in this section. Also a convenient method for ranking fuzzy numbers is presented. That is very important before fuzzy linear programming problem is defined.

Definition (1): Let R be the set of real numbers, a fuzzy subset A will be defined by its characteristic function, called the membership function, which takes its value in the interval [0, 1] instead of the binary set {0,1}

$x \in R : \mu_A(x) \in [0, 1]$

Definition (2): The α – cut of a fuzzy set A is the subset of R defined as

$A_\alpha = \{x \in R \setminus \mu_A \geq \alpha\}$

Definition (3): A fuzzy subset $A \in R$ is convex if and only if

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$x_1, x_2 \in R \quad \mu_A[\lambda x_1 + (1-\lambda)x_2] \geq \mu_A(x_1) \wedge \mu_A(x_2)$
$\lambda \in [0, 1]$

Definition (4): A fuzzy subset A ϵ R is normal if and only if
$x \in R \quad \vee_x \ \mu_A(x) = 1$

Definition (5): A fuzzy number is a fuzzy subset that is convex and normal.

There are many different types of special fuzzy numbers; here we will focus our attention on triangular fuzzy numbers as they will be used in forming our fuzzy linear programming problem.

Definition (6): A fuzzy number A is said to be a triangular fuzzy number denoted by (a_1, a_2, a_3) if its membership function μ_A is given by

$$\mu_A(x) = \begin{cases} (x - a_1) / (a_2 - a_1) & a_1 \leq x \leq a_2 \\ 1 & x = a_2 \\ (a_3 - x) / (a_3 - a_2) & a_2 \leq x \leq a_3 \end{cases}$$

Let A and B are two triangular fuzzy numbers:
$A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ then
(i)$A + B = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$
(ii)$A - B = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$
(iii)$A \times B \approx (c_1, c_2, c_3)$ where
$c_1 = \text{minimum } (a_1b_1, a_1b_3, a_3b_1, a_3b_3)$
$c_2 = a_2b_2$
$c_3 = \text{maximum } (a_1b_1, a_1b_3, a_3b_1, a_3b_3)$

(iv) $$\beta A = \begin{cases} (\beta a_1, \beta a_2, \beta a_3) & \text{for } \beta > 0 \\ & \beta \in R \\ (\beta a_3, \beta a_2, \beta a_3) & \text{for } \beta < 0 \end{cases}$$

A powerful method for comparing fuzzy numbers is by using a defuzzification function. If F(R) is the set of all fuzzy numbers defined on R the set of real numbers then a defuzzification function $G : F(R) \rightarrow R$ maps each fuzzy number into a real ordinary number where there is a natural order. The order rules are then as follows:
A > B if and only if G(A) > G(B)
A = B if and only if G(A) = G(B)
A < B if and only if G(A) < G(B)

Where A and B are two fuzzy numbers belong to F(R). A lot of work was done by researcher to develop the most suitable defuzzification function. Yager (1981) has proposed several functions based on the center of gravity or the possibility theory. Chang (1981) and Adamo (1980) use the concept of α – level set to formulize their functions. Ramik (1985) introduced two approaches: a single value approach and a multi valued approach.

Fuzzy linear programming:

A fuzzy linear programming problem is defined as follows:
Max. $\tilde{Z} = C\tilde{X}$
Subject to $A\tilde{X} \geq \tilde{B}$
$\tilde{X} \geq 0$
Where $\tilde{B} = (\tilde{b}_j)_{m\times 1}$, $\tilde{X} = (\tilde{x}_j)_{n\times 1}$, $A = (a_{ij})_{m\times n}$, $C = (\tilde{c}_j)_{1\times n}$

A fuzzy vector $\tilde{X} \geq 0$ is said to be a fuzzy feasible solution for FLP problem if it satisfies the constraints $A\tilde{X} \geq \tilde{B}$, and a fuzzy feasible solution $\tilde{X}^*$ is called a fuzzy optimal solution for the problem if for all fuzzy feasible solutions, we have $C\tilde{X}^* \geq C\tilde{X}$.

In real life problems the decision maker is faced by uncertainty in many different ways. Introducing fuzzy sets theory to the world of operations research makes the problem formulation much more convenient and representing the real situation. However even after formulating the linear programming problem as a fuzzy linear programming problem, one cannot be sure that all the entered values will remain the same for a long time. Factors like cost, time or availability of product etc. change widely. In this case sensitivity analysis for the fuzzy linear programming problem needs to be applied.

Sensitivity analysis:

Suppose that the simplex method produces an optimal solution for the given FLP problem. Now we are going to use the optimality conditions in order to find a new optimal solution if some of the data change without resolving the problem from the beginning. The following cases are considered:

Case 1. Deletion of a fuzzy variable,
Case 2. Addition of a fuzzy variable,
Case 3. Deletion of a fuzzy constraint,
Case 4. Addition of a fuzzy constraint.

Deletion of a fuzzy variable:

Obviously if a non-basic fuzzy variable or a basic one at zero level is to be deleted, there will be no change in the fuzzy optimal solution obtained. However deleting a fuzzy variable at positive level means converting it into a non-basic variable. We then have to do the following:

i- Remove the entire column of the variable to be deleted from the optimal table.
ii- Multiply the entire row of this variable by (-1).
iii- Apply fuzzy dual simplex method which will include the removal of this variable.

Addition of a fuzzy variable:

Adding a new fuzzy variable may disturb the optimality obtained. Suppose that a new item $\tilde{x}_{n+1}$ is considered for possible production with corresponding objective function coefficient c_{n+1} and corresponding constraints column A_{n+1}. The new problem will be

Max. $\tilde{Z} = C\tilde{X} + c_{n+1}\tilde{x}_{n+1}$

Subject to $A\,\tilde{X} + A_{n+1}\tilde{x}_{n+1} \geq \tilde{B}$

$\tilde{X} \geq 0, \tilde{x}_{n+1} \geq 0$

Where $\tilde{B} = (\tilde{b}_j)_{m\times 1}$,$\tilde{X} = (\tilde{x}_j)_{n\times 1}$,$A = (a_{ij})_{m\times n}$, $C = (\tilde{c}_j)_{1\times n}$

Without resolving the problem, it is easy to determine whether producing this new item worthwhile or not. Then the following has to be done:

i- Put $\tilde{x}_{n+1}$ = 0, then ($\tilde{X}^*$, 0) is a fuzzy feasible solution to the new problem.
ii- Check the relative objective function to determine whether ($\tilde{X}^*$, 0) is the optimal solution or $\tilde{x}_{n+1}$ has to be added in the basis, so iterations of fuzzy simplex method will be applied till an optimal solution is obtained.

Deletion of a fuzzy constraint:

If the fuzzy constraint to be deleted is satisfied in the interior region of the feasible solutions then its deletion will not affect the obtained optimal solution. However deleting a fuzzy constraint which is satisfied on the boundary of the feasible solutions region may cause change in the optimal solution so this case must be checked.

Addition of a fuzzy constraint:

If due to new conditions of the production process a new constraint has to be added to the problem, we will have two cases. If the obtained optimal solution satisfies the new constraint, obviously it is also an optimal solution to the new problem. But if it does not satisfy the new added constrains, we have to find a new optimal solution that is found in the new region of feasible solutions.

As an application of the previous cases we are now going to study a famous type of LP problems: *The Transportation Problem.*

The transportation problem:

LP can be applied to a wide variety of optimization problems. Some certain types of optimization problems are worth studying separately. One of these types is network flow problems. These typically related to applications involving transportation, assignment and transshipment models. The transportation problem arises in planning for the distribution of goods and services from several supply locations (called origins) to several demand locations (called destinations). The usual objective is to minimize the cost of shipping goods from origins to destinations. The problem formulation for n origins and m destinations will be as follows:

$$\text{Min } \tilde{Z} = \sum_{i=1}^{n}\sum_{j=1}^{m} C_{ij}\ \tilde{X}_{ij}$$

$$\sum_{j=1}^{m} \tilde{X}_{ij} \leq \tilde{S}_i \qquad i = 1,2,3,\ldots\ldots..,n \qquad \text{supply}$$

$$\sum_{i=1}^{n} \tilde{X}_{ij} = \tilde{d}_j \qquad j = 1,2,3,\ldots\ldots..,m \qquad \text{demand}$$

$$\tilde{X}_{ij} \geq 0 \qquad \text{for all } i,j$$

Where $\tilde{X}_{ij}$ denotes the amount shipped from the i-th origin to the j-th destination, C_{ij} denotes the cost of shipping $\tilde{X}_{ij}$ per unit, $\tilde{S}_i$ is the total supply produced by origin i, $\tilde{d}_j$ is the demanded amount required by destination j. The transportation problem always has a bounded feasible solution unless the demand exceeds the supply.

Numerical example:

A company has 3 different plants for producing a certain item, the monthly production capacity of each of them is shown in table (1). It is represented by a triangular fuzzy number as the decision maker cannot set a fixed value. That is the real life situation due to work conditions (maintenance of some machines, outbreak of some workers, increase of number of workers during universities holidays…).

Table 1: Production capacity of each plant.

Origin	Production Capacity (units)
1	(4600, 5000, 5200)
2	(5500, 6000, 6200)
3	(2400, 2500, 2600)

There are 4 distribution centers in different places; the monthly demand of each of them is represented by a triangular fuzzy number in table (2). These fluctuations in the demands are related to the clients' mode and the nature of the product.

Table 2: Demand of each distribution center.

Destination	Demand (units)
1	(5800, 6000, 6100)
2	(3800, 4000, 4200)
3	(1700, 2000, 2100)
4	(1200, 1500, 1600)

Obviously 12 distribution routes can be used; the transportation cost per unit for each of them is presented in table (3).

Table 3: Transportation cost per unit for each distribution route.

	Transportation Cost (per unit) Destination			
Origin	1	2	3	4
1	3	2	7	6
2	7	5	2	3
3	2	5	4	5

Combining the objective with the supply and demand conditions lead to a LP problem with 12 variables and 7 constraints;

Min. $\tilde{Z}$=3 $\tilde{X}_{11} + 2\tilde{X}_{12} + 7\tilde{X}_{13} + 6\tilde{X}_{14} + 7\tilde{X}_{21} + 5\tilde{X}_{22} + 2\tilde{X}_{23} + 3\tilde{X}_{24} + 2\tilde{X}_{31} + 5\tilde{X}_{32} + 4\tilde{X}_{33} + 5\tilde{X}_{34}$

Subject to

$$\tilde{X}_{11} + \tilde{X}_{12} + \tilde{X}_{13} + \tilde{X}_{14} \leq (4600,5000,5200)$$

$$\tilde{X}_{21} + \tilde{X}_{22} + \tilde{X}_{23} + \tilde{X}_{24} \leq (5500,6000,6200)$$

$$\tilde{X}_{31} + \tilde{X}_{32} + \tilde{X}_{33} + \tilde{X}_{34} \leq (2400,2500,2600)$$

$$\tilde{X}_{11} + \tilde{X}_{21} + \tilde{X}_{31} = (5800,6000,6100)$$

$$\tilde{X}_{12} + \tilde{X}_{22} + \tilde{X}_{32} = (3800,4000,4200)$$

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$$\tilde{X}_{13} + \tilde{X}_{23} + \tilde{X}_{33} = (1700, 2000, 2100)$$
$$\tilde{X}_{14} + \tilde{X}_{24} + \tilde{X}_{34} = (1200, 1500, 1600)$$
$$\tilde{X}_{ij} \geq 0 \quad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4$$

The optimal solution of this LP problem is presented in table (4).

Table 4: Optimal solution.

$\tilde{X}_{11}$	(3400, 3500, 3600)	$\tilde{X}_{23}$	(1700, 2000, 2100)
$\tilde{X}_{12}$	(1200, 1500, 1700)	$\tilde{X}_{24}$	(1200, 1500, 1600)
$\tilde{X}_{13}$	(0, 0, 0)	$\tilde{X}_{31}$	(2400, 2500, 2600)
$\tilde{X}_{14}$	(0, 0, 0)	$\tilde{X}_{32}$	(0, 0, 0)
$\tilde{X}_{21}$	(0, 0, 0)	$\tilde{X}_{33}$	(0, 0, 0)
$\tilde{X}_{22}$	(2300, 2500, 2600)	$\tilde{X}_{34}$	(0, 0, 0)

The corresponding value of the objective function is $\tilde{Z}$ =(35900, 39500, 41400)

After solving the problem the decision maker may face one or more of the following situations and he has to adapt his optimal solution to fit the new circumstances:

(1) A certain route may be temporarily deleted (due maintenance of the road, cutting of the road by some people………) that means a fuzzy variable has to be deleted. If this fuzzy variable is non-basic, its deletion will not affect the fuzzy optimal solution obtained. For example if we delete $\tilde{X}_{13}$ which represents the goods shipped from origin 1 to destination 3, the optimal solution still the same. However deleting a basic variable will affect the obtained optimal solution. For example if we delete $\tilde{X}_{24}$ which represents the goods shipped from origin 2 to destination 4, removing the column associated to $\tilde{X}_{24}$ and multiplying all the entries in its row by (-1). Knowing that the feasibility has been disturbed as a basic variable is now at zero level, we have to restore the feasibility. Therefore the following optimal solution will be obtained:

Table 5: Optimal solution after deleting $\tilde{X}_{24}$

$\tilde{X}_{11}$	(3400, 3500, 3600)	$\tilde{X}_{23}$	(1700, 2000, 2100)
$\tilde{X}_{12}$	(0, 0, 0)	$\tilde{X}_{24}$	deleted
$\tilde{X}_{13}$	(0, 0, 0)	$\tilde{X}_{31}$	(2400, 2500, 2600)
$\tilde{X}_{14}$	(1400, 1500, 1700)	$\tilde{X}_{32}$	(0, 0, 0)
$\tilde{X}_{21}$	(0, 0, 0)	$\tilde{X}_{33}$	(0, 0, 0)
$\tilde{X}_{22}$	(3800,4000,4100)	$\tilde{X}_{34}$	(0, 0, 0)

The results obtained in table (5) gives $\tilde{Z}$ = (45800, 48500, 50900). The change in the optimal values can easily be observed. For example the variable $\tilde{X}_{14}$ turns to a basic variable to compensate the decrease in the shipped goods to destination 4 due to the deletion of variable $\tilde{X}_{24}$.

(2) A new route may be established after studying the LP problem, so this new route has to be added as a new fuzzy variable. Optimality may be disturbed due to the addition of some columns in the optimal table. For example a new route will added connecting origin 1 to destination 1 with transportation cost 2 per unit (less than the cost in the existing route connecting the same origin to the same destination) but this new route's capacity is limited as it should be less than about 2500 units per month. Let this new route be represented by the fuzzy variable $\tilde{Y}_{11}$ which will appear in both the objective as well as the constraints. Starting by setting this new variable equals zero and checking the additional relative cost, we find that $\tilde{Y}_{11}$ has to be added to the basis. The new optimal solution is presented in table (6).

Table 6: Optimal solution after adding $\tilde{Y}_{11}$.

$\tilde{X}_{11}$	(2400, 2500, 2600)	$\tilde{X}_{23}$	(1700, 2000, 2100)
$\tilde{X}_{12}$	(0, 0, 0)	$\tilde{X}_{24}$	(1200, 1500, 1600)
$\tilde{X}_{13}$	(0, 0, 0)	$\tilde{X}_{31}$	(900, 1000, 1200)
$\tilde{X}_{14}$	(0, 0, 0)	$\tilde{X}_{32}$	(1200, 1500, 1600)
$\tilde{X}_{21}$	(0, 0, 0)	$\tilde{X}_{33}$	(0, 0, 0)
$\tilde{X}_{22}$	(2300, 2500, 2600)	$\tilde{X}_{34}$	(0, 0, 0)
		$\tilde{Y}_{11}$	(2300, 2400, 2500)

With a total cost $\tilde{Z}$ = (38100, 42800, 45200). It is clear that the optimal solution has been changed as the goods have found a new route with low cost to pass through. That also affects the other routes related to origin 1 or destination 1. In studying this case we have not only added a new fuzzy variable but we also added a new constraint that disturbs the fuzzy simplex format so the feasibility disturbed.

(3) A certain production plant is temporarily closed (May this happens because of a safety problem or workers' outbreak……..). This means that a certain origin has to be deleted with its related constraints. For example let origin 3 be the one that shuts for a period of time. In other words variables $\tilde{X}_{31}$, $\tilde{X}_{32}$, $\tilde{X}_{33}$ and $\tilde{X}_{34}$ as well as the 3rd constraint have to be deleted. As this constraint is satisfied on the boundary, it will cause a change in the obtained fuzzy solution. In other words the demand centers must satisfy its needs through other routes instead of the deleted ones. But since in the original LP problem the demands and the supplies are equal, deleting origin 3 makes the demands exceed the supplies. Then the model is infeasible. The decision maker has to suggest increasing the production capacities of the working plants to meet the demands of the distribution centers.

Conclusion:

Some important cases for the sensitivity analysis of fuzzy linear programming are discussed. Applying these cases on the transportation problem helps the decision maker to be more flexible against perturbations that arise after solving the problem. We hope to extend this study to other engineering applications.

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