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A Note on Cone Metric Spaces

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ABSTRACT

The Purpose of this document is to accept the generalization of T- contractive type mapping on complete metric spaces. On this paper, we obtained sufficient conditions for the existence of a unique fixed point of generalized contractive type mappings on complete cone metric spaces depending on another function.

Keywords: Cone metric spaces, normal Cone metric, expansive mapping, fixed- point.

1. Introduction

Mustafa and Sims, (2006) considered a more generalization of metric spaces, that of G- metric spaces. Mustafa and Awawdeh, (2008) presented several fixed-point theorems for mapping satisfying different contractive conditions in G- metric spaces. Long- Guang and Xian, (2007) generalized some facts of metric spaces, a cone metric space. Moreover, they described the convergence of sequences in cone metric spaces and introduced the corresponding somethings of completeness. After wards, they presented some fixed-point theorems of contractive mappings on complete cone metric spaces. Some of the mentioned results were considered by Rezapour and Hambarani, (2008) omitting the assumption of normality on the cone. Beiranvand *et al.* (2009) introduced category of T-Contraction and T−Contractive functions, extending the Banach contraction principle. The results are generalized by Beg *et al.* (2010) with normal constant K=1. In addition, we have considered the existence of fixed-points for non – explosive map in cone metric space see (Raja and Vaezapour, 2008; Zhang, 2010). Long- Guang and Zhang, (2007) in obtained some fixed-point theorems for mappings satisfying different contractive conditions. And they generalized some Fixed-point theorems in metric spaces. Moreover, they studied the existence and uniqueness of the Fixed- point for pair of expansive mapping defined on a complete metric space. We will present some important results of fixed-point theorems of contractive mappings on cone metric spaces. The Purpose of this paper is to analyze the existence and uniqueness of Fixed- point, we generalized the results about some fixed points of Raja and Vaezapour, (2008). Moreover, we tried to trace some important results of Fixed – point theory,

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common Fixed - point theorems in cone metric spaces, and finally some coupled Fixed-point theorems in cone metric spaces.

2- Preliminary notes:

Concisely, we give some basic definitions and concepts, which can be considered as a background to this work.

Cones: 2.1

In this section, we shall define the concept of cones and show some examples.

Definition 1: Mustafa and Awawdeh, (2008).

Let E always be a real Banach space. $and\ P$ a subset of A is called a cone in A if satisfies the following:

(i) $P\ is\ non\ \ closed, convex, nonempt, and\ P\ \neq\ \{0\}$; where 0 is the zero vector of P

(ii) $If\ a, b\ \in\ R, a, b\ \geq\ 0, and\ \ x, y\ \in\ P\ imply\ that\ ax\ +\ by\ \in\ P$;

$(iii) x\ \in\ P\ and\ -x\ \in\ P\ imply\ that\ x\ =\ 0.$

Definition 2: Mustafa and Awawdeh, (2008).

Given any cone P ⊂ A, we define a partial ordering ≤ with respect to P on E as:

1- $x \leq\ y$, if and only if, $y\ - x\ \in\ P$.

2- x < y to indicate that $x \leq\ y$ but $x \neq\ y$.

3- $x << \ y\ where\ y - \ x\ \in P^0.\ where\ P^0\ denotes\ the\ interior\ of\ P.$

Definition 3: Mustafa and Awawdeh, (2008).

A cone P is called normal if there is a number K > 0, such that for all $x, y \in E, 0 \leq\ x \leq\ y$

$implies\ \|x\ \| \leq k\|\ y\|\ , the\ least\ positive\ number\ k\ satisfying\ above\ inequality$

is called the normal constant of P.

Some family of examples.

Examples of cones:

Example 1: Mustafa and Sims, (2006).

$.Let\ A =\ R^2\ , then\ P =\ \{(x_1, x_2) \in A\ ;\ x_i\ \geq 0; \forall\ i = 1, 2\}\ is\ a\ cone$

Example 2: Rezapour and Hambarani, (2008).

$.Let\ A =\ R^n\ , then\ P =\ \{(x_1, x_2 \dots \dots, x_n) \in A\ \ ;\ x_i\ \geq 0; \forall\ i = 1,2 \dots \dots, n\}\ is\ a\ cone$

Cone metric spaces:- **2.2**

In this portion, we will define cone metric spaces, and show that some theorems and properties about cone metric, besides having discussed the relation between metric spaces and cone metric spaces.

Let A to be a real Banach spaces, and P be a cone of A. We define cone metric spaces as follows.

Definition 4: Mustafa and Sims, (2006)

Let X be a nonempty set. Suppose the mapping $d : X \times X \rightarrow A\ satisfies$:

$(d1)\ 0 < d(x, y) for\ all\ x, y \in X\ and\ d(x, y) = 0\ if\ and\ only\ if\ x = y$;

$(d2)\ d(x, y) = d(y, x)\ , for\ all\ x, y \in X$;

$(d3)\ d(x, y) \leq d(x, z) + d(y, z)\ \ for\ all\ x, y, z \in X$.

$Then\ d\ is\ called\ a\ cone\ metric\ on\ X. and\ (X, d\ ) is\ called\ a\ cone\ metric\ space$ ·

Note from the definition that cone metric spaces generalize metric spaces.

Here are some familiar examples of cone metric spaces.

Example 3: Mustafa and Awawdeh, (2008)

$Let\ A = R^3\ with\ P = \{(x, y, z): x, y, z \geq 0\}, X = R$.

$and\ D : X \times X \times X \longrightarrow R^3\ such\ that$:

$$D(x. y) = \{|x - y|, \alpha_1|y - z|, \alpha_2|x - z|\}$$

$where\ \alpha_1, \alpha_2 > 0\ . Then\ (X, D)$ is $3 - \text{dim}$ cone metric space.

Example 4: Mustafa and Awawdeh, (2008) $Let\ A = R^n\ with\ P = \{(x_1, x_2 .., x_n): x_1 \geq 0 , \forall i = 1, ..., n\}\quad X = R^n$.

$and\ D = X \times X \longrightarrow E\ such\ that$:

$$D(x. y) = \{|x - y|, \alpha_1|x - y|, ..., \alpha_{n-1}|x - y|\}$$

$where\ \alpha_i > 0\ for\ all\ 1 \leq i \leq n - 1. Then\ (X, D)$ is a cone metric space ·

Definition 5: Beg *et al.* (2010)

$Let\ (X, d) be\ a\ cone\ metric\ space\ and\ A \subseteq X$ $\cdot Then\ A\ is\ said\ to\ be\ bounded$

$above\ if\ \exists e \in E;\ e \succcurlyeq 0\ such\ that, d\ (x, y) \preccurlyeq e$;

$\forall x, y \in A, and\ A\ is\ called\ bounded\ \ if \delta(A) = \sup\{d(x, y): x, y \in A\}\ exists\ in\ E$ ·

2.3-Convergence in Cone Metric Spaces:

Below, we present the notion of convergence of sequences in cone metric spaces.

Definition 6: Beg *et al.* (2010)

$Suppose\ that\ (X, d)\ be\ a\ cone\ metric\ space, x \in X\ and\ \{x_n\}_{n\geq 1}\ a\ sequence\ in\ X, Then$:

$i)\ \{x_n\}_{n\geq 1} converges\ to\ x, wheneve\ \forall c \in A\ with\ 0 \leq c\ \ , there\ is\ a\ natural\ number\ N$.

$such\ that\ d(x_n, x) \preccurlyeq c, \forall\ n \geq N. We\ denote\ this\ by\ \lim_{n\to\infty} x_n = x\ or\ x_n \rightarrow x$.

$ii)\ \{x_n\}_{n\geq 1} is\ Cauchy\ sequence\ whenever\ for\ every\ c \in A\ with$

$0 \preccurlyeq c\ \ , there\ is\ a\ natural\ number\ N. such\ that\ d(x_n, x_m) \preccurlyeq c, \forall n, m \geq N$.

$iii)\ (X,d)$ *is complete cone metric space if every Cauchy sequence is convergent in* X.

Lemma 1: Beg *et al.* (2010)

Let (X,d) *be a cone metric space,* P *is a normal cone with normal constant* k

and $\{x_n\}$ *is a sequence in* X, *then*

$i)\ \{x_n\}$ *converges to* $x \Leftrightarrow \lim_{n\to\infty} d(x_n,x) = 0$.

$ii)$ *If* $\{x_n\}$ *is convergent, then it is Cauchy sequence.*

$iii)\ \{x_n\}$ *is a Cauchy sequence if, and only if,* $\lim_{n.m\to\infty} d(x_n,x_m) = 0$.

$iv)$ *If* $\{x_n\} \to x$ *and* $\{x_n\} \to y$, *as* $n \to \infty$, *then* $x = y$.

$v)$ *If* $\{x_n\} \to x$, *as* $n \to \infty$ *and* $\{y_n\}$ *is another sequence in* X *such that* $\{y_n\} \to y$

then $d(x_n,y_n) \to d(x,y)$.

Definition 7: (Dubey *et al.,* 2013)

Let (X, d) be a cone metric space. P a normal cone with normal constant k

and $T : X \to x_n$. Then T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that

$\lim_{n\to\infty} Tx_n = Tx$ for every $\{x_n\}$ in X . T is said to be sequentially convergent if we have.

for every sequence $\{y_n\}$, when $\{T(y_n)\}$is convergent, then$(y_n)$is also convergent.

Definition 8: Dubey *et al.,* (2013)

Let (X, d) be a cone metric space. if for any sequence in X there is a subsequence

$\{x_{n_i}\}$ of $\{x_n\}$ such that. $\{x_{n_i}\}$ is convergent in X. Then X is called a sequentially compact

cone metric space.

Now, it is time to define 3- dimensional cone metric spaces.

Definition 9:

Let X be a nonempty set. Suppose *the* mapping $d : X \times X \times X \to A$ *satisfies*

$(d1)\ 0 < d(x,y,z)$ *for every* $x,y,z \in X$ *and* $d(x,y,z) = 0$ *if and only if* $x = y = z$;

$(d2)\ d(x,y,z) = d(y,x,z) = \cdots \ldots \ldots$ *for every* $x,y,z \in X$;

$(d3)\ d(x,y,z) \le d(x,y,w) + d(w,y,z)$ *for every* $x,y,z,w \in X$.

Then d is called a cone metric in X, and (X, d) is called 3- dimensional cone metric space.

2.4. G- cone metric spaces:

Definition 10: Beg *et al.* (2010)

Let $X \neq \emptyset$. *Suppose the mapping* $G : X \times X \times X \to A$ *satisfies*:

$(G1)\ G(x,y,z) = 0$ *if* $x = y = z$.

$(G2)\ 0 < G(x,x,y);\ whenever\ x \neq y, for\ every\ x,y \in X.$

$(G3) G(x,x,y) \leq G(x,y,z);\ whenever\ y \neq z.$

$(G4) G(x,y,z) = G(x,z,y) = G(y,x,z) = \ldots (Symmetric\ in\ all\ three\ variables).$

$(G5) G(x,y,z) \leq G(x,a,a) + G(a,y,z) for\ all\ x,y,z,a \in X.$

Then G is called a generalized cone metric in X, and X is called a generalized cone metric space or more specifically a $G -$ cone metric space.

The concept of a $G -$ cone metric spaces is more general than that of a $G -$ metric spaces and cone metric spaces. For the definition of $G -$ metric, cone metric spaces and related concepts we refer the reader to Rezapour and Hambarani, (2008); Zhang, 2010; Azam *et al.,* (2010) Beg *et al.* (2010)

Definition 11: Sastry *et al.* (2011)

$A\ Generalized\ cone\ metric\ space\ X\ is\ symmetric\ if\ G(x,y,y) = G(y,x,x).$

$for\ all\ x,y \in X \cdot$

Examples of Generalized metric spaces.

Example 5: Sastry *et al.* (2011)

$Let\ (X.G)\ be\ a\ cone\ metric\ space, consider\ G : X \times X \times X \rightarrow A.$

$by\quad G(x,y,z) = d(x,y) + d(y,z) + d(z,x). Then\ G\ is\ a\ generalized\ metric\ space.$

Example 6: Sastry *et al.* (2011)

$Let\ X = \{a.b\}, A = R^3, P = \{(x,y,z) \in E\ |x,y,z \geq 0\}.$

$Define\ G : X \times X \times X \rightarrow A\ by\ G(a,a,a) = (0,0,0) = G(b,b,b).$

$G(a,b,b) = (0,1,1) = G(b,a,b) = G(b,b,a).$

$G(b,a,a) = (0,1,0) = G(a,b,a) = G(a,a,b)$

$Note\ that\ X\ is\ non-symmetric\ G-cone\ metric\ space\ as\ G(a,a,b) \neq G(a,b,b)$

Proposition 1: Sastry *et al.* (2011)

$Let\ (X,d_G)\ be\ a\ generalized\ cone\ metric\ space. define\ d_G : X \times X \rightarrow E.$

$by\quad d_G(x,y) = G(x,y,y) + G(y,x,x), then\ (X,d_G)$ is a cone metric spaces

Proposition 2: Sastry *et al.* (2011); Turkoglo *et al*., 2012)

Let X be a generalized cone metric space then the following are equivalent.

(i) $\{x_n\}$ is converges to x.

(ii) $G(x_n,x_n,x) \rightarrow 0, as\ n \rightarrow \infty.$

$(iii) G(x_n,x,x) \rightarrow 0, as\ n \rightarrow \infty$

$(iv)\ G(x_m,x_n,x) \rightarrow 0, as\ m.n \rightarrow \infty.$

3. Main results:

Next, we are going to generalize in the case when we have n- continuous and onto mapping $f_1, f_2. \dots \dots, f_m$ where $f_i : X \to X$, $1 \le i \le m$. satisfying the following condition.

$$d\left(f_r(x). f_t(x)\right) \ge K\, d(x. y) + L\, d\left(x. f_r(x)\right) + Md\left(y. f_t(y)\right)$$

For all $x. y \in X$. $K \ge -1$, $L \ge 1$, $M < 1$ *are constant with* $K + L + M > 1$.

The following fact is to consider and prove the above condition that implies the finite number of continuous and onto mapping having common a fixed-point in a complete cone metric space.

Theorem 1:

Let(X, d) be a complete cone metric space and suppose $f_1, f_2. \dots \dots, f_m : X \to X$ be any continuous and onto mappings satisfying the condition

$$d\left(f_r(x), f_t(x)\right) \ge K\, d(x, y) + L\, d\left(x, f_r(x)\right) + Md\left(y, f_t(y)\right)$$

For all $x, y \in X$, $K \ge -1, L \ge 1, M < 1$ *are onstant with* $K + L + M > 1$. .

Then $f_1, f_2. \dots \dots, f_m$ have a common fixed point in X.

Proof:

Suppose that $x_0 \in$ X, since $f_1, f_2. \dots \dots$, f_m are onto, there exist $x_i \in$ X, i = 1, 2,....., m such that

$f_1(x_1) = x_0, f_2(x_2) = x_1. \dots \dots \dots, f_m(x_m) = x_{m-1}$.

We define x_n, *and* x_{n+1} *by* $x_n = f_1(x_{n+1})$, $x_{n+1} = f_2(x_{n+2})$ *for* $n = 0,1 \dots \dots \dots$

If $x_n = x_{n+1}$ *for some* $n \ge 1$, *then* x_n *is fixed point of* f_1 *and* f_2.

Put x = $x_{n+1}, y = x_{n+2}$, *we have*

$$d\,(x_n, x_{n+1}) = d\,(f_1 x_{n+1}, f_1 x_{n+1})$$

$$d\,(x_n, x_{n+1}) \ge K\, d(x_{n+1}, x_{n+2}) + L\, d\left(x_{n+1}, f_1(x_{n+1})\right) + Md\left(x_{n+2}, f_2(x_{n+2})\right)$$

$$\ge K\, d(x_{n+1}, x_{n+2}) + L\, d(x_{n+1}, x_n) + M\, d(x_{n+2}, x_{n+1})$$

$(1 - L)\, d\,(x_n, x_{n+1}) \ge (K + M)\, d\,(x_{n+1}, x_{n+2})$

$$d\,(x_n, x_{n+1}) \ge \frac{(K + M)}{(1 - L)}\, d\,(x_{n+1}, x_{n+2})$$

$d\,(x_n, x_{n+1}) \ge h\, d\,(x_{n+1}, x_{n+2})$

Where $h = \frac{K+M}{1-L}$, $0 \le h \le 1$

In general
$d\,(x_n, x_{n+1}) \le h\, d(x_{n-1}, x_n) + \cdots + h^n d(x_0, x_1)$

So for $n < k$, $d\,(x_n, x_k) \le d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m)$

$$\le [\, h^n + h^{n+1} + \cdots + h^{m-1}]\, d(\, x_0,\, x_1)$$

$\le \frac{h^n}{1-h}\, d(\, x_0,\, x_1)$

Let c ≥ 0 be given, let a natural number N_1 be Such that h^n/ (1-h) d (x_0, x_1) ≤ c. For all

$n \geq N_1$ · Thus $d(x_n, x_m) \leq c$, $for\ n < m$ ·

Therefore $\{x_n\}$ is a Cauchy sequence in (X, d).Since(X, d) is a complete cone metric space, there exists $x^* \in X$ such that $x_n \to x^*\ as\ n \to \infty$ ·

If f is a continuous, then

$d\ (\mathrm{f_r}(x^*), x^*) \leq d\big(\mathrm{f_r}(x_{n+1}), \mathrm{f_r}(x^*)\big) + Md(\mathrm{f_r}(x_{n+1}), x^*) \quad \to 0\ as\ n \to \infty$

Since $x_n \to x^*\ and\ \mathrm{f_r}(x_{n+1}) \to \mathrm{f_r}(x^*)\ as\ n \to \infty$, $r = 0.1 \dots \dots m$ ·

Therefore $d\ (\mathrm{f_r}(x^*), x^*) = 0, which\ this\ imples\ that\ \mathrm{f_r}(x^*) = x^*$.

Hence x^* is fixed point.

Next, we consider two conditions on some map on a complete cone metric space which implies that maps has a unique fixed-point.

Theorm 2 :

Let (X, d) be a complete cone metric space,P a normal cone with a normal constant K. Let $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ satisfy the following conditions

$$d\ (Tx, Ty) \leq K\{ d(x, y) + d(Ty, x)\}$$

Or
$$d\ (Tx, Ty) \leq K\{ d(x, y) + d(Tx, y)\}$$

For all $x, y \in X$, where $K \in [0,1[$ is a constant. Then T has a unique fixed point in$X$.

Proof:

Suppose that

$$d\ (Tx, Ty) \leq K\{ d(x, y) + d(Ty, x)\}$$

and let $x_0 \in \mathrm{X}$ and $x_1 \in X$, such that $x_1 = \mathrm{T}\, x_0$ and $x_{n+1} = \mathrm{T}\ x_n = T^{n+1}\, x_0$ for n=1, 2 … We have,

d (x,y) = d (x_{n+1}, x_n) = d(T x_n, T $x_{n-1})$, then using condition

d (T x_n, T $x_{n-1}) \leq K\{$ d($x_n, x_{n-1})$ + d (T $x_{n-1}, x_n)\}$.

So, d (T x_n, T $x_{n-1}) \leq K\{$ d ($x_n, x_{n-1})$ + d ($x_n, x_n)\}$.

d ($x_{n+1}, x_n) \leq K$ d ($x_n, x_{n-1})$, (1)

$we\ have$ For n > m,

d ($x_n, x_m) \leq$ d ($x_n, x_{n-1})$ + d($x_{n-1}, x_{n-2})$ +⋯+ d ($x_{m+1}, x_m)$.

From condition (1).

d ($x_{n+1}, x_n) \leq K$ d ($x_{n-1}, x_{n-2})$.

$\leq K^2$ d ($x_{n-2}, x_{n-3})$.

$\leq K^{n-1}$ d ($x_1, x_0)$.

d ($x_n, x_m) \leq [\, K^{n-1} + K^{n-2} + \cdots + K^m]$ d($x_1, x_0)$.

$\leq \frac{K^n}{1-K}$ d ($x_1, x_0)$.

Since, $\|d\,(x_n, x_m)\| \le \frac{K^n}{1-K}\|d\,(x_1, x_0)\|$.

This implies $d\,(x_n, x_m) \to 0 \quad as\, n, m \to \infty$.

Choose a natural number N_1 ,such $\frac{K^n}{1-K}$ $\|d\,(x_1, x_0)\| \le$c, for all m $\in N_1$. Thus, $d\,(x_n, x_m) \le$ c, for n > m. Hence $\{x_n\}$ is a Cauchy sequence.

By completeness of X, we have $x_n \to x^* \; as\; n \to \infty, for\; some\; x^*\; in\; X$.

Since, $d\,(Tx^*, x^*) \le d\,(T x_n, T x^*) + d(T x_n, x^*)$.

$\le K\,[\,d\,(x_n, x^*) + d(T x_n, x^*)] + d(T x_n, x^*)$.

. $\le K\,[\,d\,(x_n, x^*) + d(T x_n, x^*)] + d(T x_n, x^*)$

. $d\,(Tx^*, x^*) \le K\, d\,(x_n, x^*) + (K+1)\; d\,(T x_n, x^*)$

$d\,(Tx^*, x^*) \le K\, d\,(x_n, x^*) + (K+1)\, d\,(x_{n+1}, x^*)]$.

$\|d\,(Tx^*, x^*)\| \le [\,K\,\|d\,(x_n, x^*)\| + (K+1)\,\|d\,(x_{n+1}, x^*)\| \to 0 \quad as\; n \to \infty\,]$.

Hence$\|d\,(Tx^*, x^*)\| = 0$, this implies $Tx^* = x^*$.

Then x^* is a fixed point of T.

To prove the uniqueness of x^* , we have if y^* is another fixed point of T, then

$d\,(x^*, y^*) = d\,(Tx^*, Ty^*)$.

$\le K[\,d\,(x^*, y^*) + d\,(y^*, Ty^*)]$.

Then, $d\,(x^*, y^*) \le 2K\, d\,(x^*, y^*)$

Hence, $d\,(x^*, y^*) = d\,(x^*, y^*)\,, and\;\; x^* = y^*$. Hence, the theorem is proved.

Next, we consider one condition on some map on a complete cone metric space which implies that it map has a unique fixed-point.

Theorem 3:

Let(X, d) be a complete cone metric space, P be a normal cone with a normal constant K.

Let $T: X \to X$ that satisfies the condition:

$$d\,(Tx, Ty) \le \alpha\, d(x, y) + \beta\, d(x, Tx) + \gamma\, d(y, Ty)$$

For all $x, y \in X$, where $\alpha, \beta, \gamma \in [0,1[$ is a constant.

Then T has a unique fixed point in X.

Proof:

There exist $x_0 \in X$ and $x_1 \in X$, such that $x_1 = T x_0$ and $x_{n+1} = T\, x_n = T^{n+1} x_0$ for n =1, 2,.....

We have, $d\,(x_{n+1}, x_n) = d\,(T x_n, T x_{n-1})$, then from condition

$d(T\, x_n, T\, x_{n-1}) \le \alpha\, d(x_n, x_{n-1}) + \beta\, k\; d(x_n, T x_n) + \gamma\, d(x_{n-1}, T\, x_{n-1})$.

So, $d(T\, x_n, T\, x_{n-1}) \le \alpha\, d(x_n, x_{n-1}) + \beta\, k\; d(x_n, x_{n+1}) + \gamma\, d(x_{n-1}, x_n)$.

$(1-\beta)$d (x_{n+1} , x_n)≤ $(\alpha+\gamma)$ d (x_n, x_{n-1}).

d (x_{n+1}, x_n) ≤ $\frac{(\alpha+\gamma)}{(1-\beta)}$ d (x_n, x_{n-1}).

d (x_{n+1}, x_n)≤ K d (x_n, x_{n-1}). (1)

Where K = $\frac{(\alpha+\gamma)}{(1-\beta)}$ For n > m,

d (x_n, x_m) ≤ d (x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) +⋯+ d (x_{m+1}, x_m).

using condition (1).

d (x_{n+1}, x_n) ≤ K d (x_{n-1} x_{n-2}) .

≤ K^2 d (x_{n-2}, x_{n-3}).

≤ K^{n-1} d (x_1, x_0).

d (x_n, x_m) ≤ [K^{n-1} + K^{n-2} +⋯ +K^m] d(x_1, x_0).

≤ $\frac{K^n}{1-K}$ d (x_1, x_0).

Since, ‖d (x_n, x_m) ‖ ≤ $\frac{K^n}{1-K}$ ‖d (x_1, x_0)‖ .

This implied d (x_n, x_m) → 0 as n, m → ∞ .

Choose a natural number N_1 such $\frac{K^n}{1-K}$ ‖d (x_1, x_0) ‖ ≤c, for all m ∈ N1. Thus, d $(x_n, x_m) \leq$ c, for n > m, hence { x_n} is a Cauchy sequence.

Since, X is complete, we have $x_n \to x^*$ as $n \to \infty$.

Since, d (Tx^*, x^*)≤ d (T x_n, T x^*) + d(T x_n, x^*).

Again using the given condition

d (Tx^*, x^*) ≤ d (T x_n, T x^*) + d(T x_n, x^*)

≤ $\alpha\, d$ (x_n,x^*) + $\beta\, d$(x_n,T x_n) + $\gamma\, d$(x^*,T x^*) + d (Tx_n,x^*)

≤ $\alpha\, d$ (x_n,x^*) + $\beta\, d$(x_n, x_{n+1}) + $\gamma\, d$(x^*,T x^*) + d (x_{n+1}, x^*)

(1-γ) d (Tx^*, x^*) ≤ $\alpha\, d$ (x_n,x^*) + $\beta\, d$(x_n ,x_{n+1}) + d (x_{n+1},x^*).

d (Tx^*, x^*) ≤ $\frac{1}{1-\gamma}$ [$\alpha\, d$ (x_n,x^*) + $\beta\, d$(x_n,x_{n+1}) + d (x_{n+1}, x^*)].

‖d (Tx^*, x^*)‖ ≤ $\frac{1}{1-\gamma}$ [α ‖d (x_n,x^*)‖ + β ‖d(x_n, x_{n+1})‖+ ‖d (x_{n+1}, x^*)‖ → 0].

Since ‖d (Tx^*, x^*)‖ = 0 , then $Tx^* = x^*$.

Then x^* is a fixed- point of T.

To prove the uniqueness, then d (x^*, y^*) = d (Tx^*, Ty^*).

≤ α d (x^*, y^*) + $\beta\, d$(x^*, Tx^*) + $\gamma\, d$ (y^*, Ty^*).

Then, d (x^*, y^*) ≤ α d (x^*, y^*).

Hence, $x^*= y^*$. Thus, the theorem is proved.

Conclusion

We have introduced new generalized cone metric spaces and proved some important properties. For our results, the concept of cone metric spaces can be considered as a fundamental tool in the Analysis Theory. We generalize gradually this notion.

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References

Azam, A., M. Arshad and I. Beg, 2010. "Banach contraction principle on cone rectangular metric spaces", Appl. Anal. Discrete. Math., 3(2): 236-241.

Beg I., M. Abbas, and T. Nazir, 2010." generalized cone metric spaces ", J. of Non-linear science and Appl., 1. 21-31.

Beiranvand, A., S. Moradi, M. Oid, H. Pazandeh, 2009" Two fixed point theorems for special mapping", arXiv: 0903. 1504 V1.

Dubey, A.K., S.K. Tiwari, R.P. Dubey, 2013." Cone Metric Spaces and Fixed Point Theorems of Generalized T- Contractive Mapping", International Journal of Pure and Applied Mathematics , 84(4): 353-363.

Long- Guang, H. and Z. Xian, 2007. "Cone metric spaces and Fixed point theorems of Contractive mappings", J. Math Anal. Appl., 332, 1468-1476.

Mustafa, Z. and B. Sims, 2006." A new approach to generalized metric spaces", J.Nonlinear and Convex Anal., 7(2): 289-297.1, 2.

Mustafa, Z., O. Hamed and F. Awawdeh, 2008. "Some fixed point theorems for mapping on complete G- metric spaces, Fixed Point Theory and Applications, Vol. 2008, Article ID 189870, 12 pages 1.

Raja, P. and S.M. Vaezapour, 2008." Some extensions of Banach contraction principle in complete cone metric spaces", Fixed Point Theory Appl., (11), Article ID 768294.

Rezapour, S., R. Hamlbarani, 2008. Some notes on the paper "Cone metric spaces and fixed point theorem of contractive mappings ", J. Math. Anal. Appl., 345, 719-724.

Sastry, K.P.R., Ch. Srinivasarao, K. Sujatha, G. Praveena and Ch. Srinivasarao, 2011." Cone Metric Spaces and Fixed Point Theorems of Generalized Contractive Mapping", International Journal of Computational Science and Mathematics, 3, (2): 133-139.

Turkoglo, D., M. Abuloha and T. Abdeljawad, 2012." Some Theorems in Cone Metric Spaces", Journal of Concrete and Applicable Mathematics, 10.

Zhang, S.S., 2010. Weak convergence theorem for Lipschizian pseudocontraction semigroups in Banach spaces ", Acta Mathematica Sinica (English Series), 26(2), 337-344.